

APPROXIMABILITY OF CONVEX BODIES AND VOLUME ENTROPY OF HILBERT GEOMETRIES

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ABSTRACT. The approximability of a convex body is a number which measures the difficulty to approximate that body by polytopes. We prove that twice the approximability is equal to the volume entropy for a Hilbert geometry in dimension two and three and that in higher dimension it is a lower bound of the entropy. As a corollary we solve the entropy upper bound conjecture in dimension three and give a new proof in dimension two from the one found in [BBV10].

INTRODUCTION AND STATEMENT OF RESULTS

Hilbert geometries are a family of metric spaces defined in the interior of an open and bounded convex set using cross-ratios following the construction of the hyperbolic geometry's projective model [Hil71]. They are actually length space with an underlying Finsler structure which is Riemannian only if the convex set is an ellipsoid [Kay67]. The present paper focuses on the volume growth of these geometries and more specifically on the volume entropy.

This immediately raises the problem of the definition of the volume one uses, as in Finsler geometry there isn't uniqueness of the notion of volume. In this paper we will mostly work with two volumes: the *Busemann volume* which also appears as the Hausdorff measure associated to the distance, and the *Holmes-Thompson volume* which is built thanks to the natural symplectic structure of the cotangent bundle. The first one is more convenient for computational purposes, but the second one possesses better and crucial properties, such as minimality property of totally geodesic hypersurface [AB09, AF98, Ber09]. Both happen to be equivalent and belong to a larger family of measures over the Hilbert geometries that we call *normalised density measures* and for which all the results presented in this paper coincide. Therefore in a open convex set Ω , we will consider a member Vol_Ω of this family of measures and if we denote by $B_\Omega(p, r)$ the metric ball of radius r centred at the point $p \in \Omega$, then the volume entropy of Ω will be defined

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as

$$(1) \quad \text{Ent}(\Omega) = \liminf_{r \rightarrow +\infty} \frac{\ln(\text{Vol}_\Omega(B_\Omega(p, r)))}{r}.$$

Let us stress out that thus defined, the entropy of Ω does not depend on either the base point p nor on the normalised density measure chosen and is actually a projective invariant attached to Ω (One can also replace the "lim inf" by a "lim sup" in all the definitions of this paper, and get the same results for the corresponding numbers).

The question we address in this essay is the *entropy upper bound conjecture* which states that if Ω is an open and bounded convex subset of \mathbb{R}^n , then $\text{Ent}(\Omega) \leq n - 1$. Let us recall the main related results.

The first one is a complete answer to the the conjecture in the two dimensional case by G. Berck & A. Bernig & C. Vernicos [BBV10]], where the authors actually obtained an upper bound as a function of d , the upper Minkowski dimension (or *ball-box* dimension) of the set of extremal points of Ω , as follows:

$$(2) \quad \text{Ent}(\Omega) \leq \frac{2}{3-d} \leq 1.$$

The second results is a more precise statement with respect to the asymptotic volume growth of balls and involves another projective invariant introduced by G. Berck & A. Bernig & C. Vernicos [BBV10] called the *centro-projective* area of Ω defined by

$$(3) \quad \mathcal{A}_p(\Omega) := \int_{\partial\Omega} \frac{\sqrt{k(x)}}{\langle n(x), x-p \rangle^{\frac{n-1}{2}}} \left(\frac{2a(x)}{1+a(x)} \right)^{\frac{n-1}{2}} dA(x),$$

where for any $x \in \partial\Omega$, $k(x)$ is the Gauss curvature, $n(x)$ the normal and $a(x) > 0$ the antipodal map defined by $p - a(x)(x - p) \in \partial\Omega$. Let us remind the reader that both k and n are defined almost everywhere as Alexandroff's theorem states [Ale39].

Now we can state the second main result of G. Berck & A. Bernig & C. Vernicos [BBV10] which encloses former results by B. Colbois and P. Verovic [CV04]. If $\partial\Omega$ is $C^{1,1}$, then

$$(4) \quad \lim_{r \rightarrow \infty} \frac{\text{Vol}_\Omega B_\Omega(p, r)}{\sinh^{n-1} r} = \frac{1}{n-1} \mathcal{A}_p(\Omega) \neq 0$$

and $\text{Ent}(\Omega) = n - 1$. Moreover, without any assumption on Ω , if $\mathcal{A}_p(\Omega) \neq 0$ then $\text{Ent}(\Omega) \geq n - 1$.

The third one which is also a rigidity results requires stronger assumptions on the Hilbert geometries: they have to be divisible, which means that they admit a compact quotient, and they have to be hyperbolic in the sense of Gromov, which implies by Y. Benoist [Ben03] that their boundary is C^1 and strictly convex. Let us stress out that among them only the hyperbolic geometry has a $C^{1,1}$ boundary, and as hyperbolicity implies non-nullity of the Cheeger constant (see B. Colbois

and C. Vernicos [CV07]), their entropy is strictly positive. M. Crampon [Cra09] results states that for a divisible open bounded convex set Ω in \mathbb{R}^n whose boundary is C^1 ,

- $\text{Ent}(\Omega) \leq n - 1$;
- and equality occurs if and only if Ω is an ellipsoid.

In the present paper we relate the entropy to another invariant associated to a convex set, its *approximability* whose name was introduced by Schneider and Wieacker [SW81] and which somewhat measure how well a convex set can be approximated by polytopes. More precisely, let $N(\varepsilon, \Omega)$ be the smallest number of vertices of a polytope whose Hausdorff distance to Ω is less than $\varepsilon > 0$. Then the approximability of Ω is

$$a(\Omega) := \liminf_{\varepsilon \rightarrow 0} \frac{\ln N(\varepsilon, \Omega)}{-\ln \varepsilon}.$$

The main result which is of interest for this essay is the following upper bound, obtained by Fejes-Toth [FT48] in dimension 2 and by Bronshteyn-Ivanov [BI76] in the general case: If Ω is a bounded open convex set in \mathbb{R}^n , then

$$(5) \quad a(\Omega) \leq \frac{n-1}{2}.$$

Our main result is the following one

Theorem (Main theorem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open convex set, then*

- $2a(\Omega) \leq \text{Ent}(\Omega)$;
- if $n = 2$ or 3 then $\text{Ent}(\Omega) = 2a(\Omega)$.

The first important corollary is that it gives a proof of the entropy upper bound conjecture in dimension 2 and 3.

Corollary (Upper entropy bound conjecture). *Let Ω be a bounded open convex set in \mathbb{R}^n , with $n = 2, 3$, then $\text{Ent}(\Omega) \leq n - 1$.*

The second one tells us that in dimension 2 there are Hilbert geometries with intermediate volume growth.

Corollary (Intermediate growth). *Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an increasing function such that*

$$\liminf_{r \rightarrow +\infty} \frac{e^r}{f(r)} > 0$$

then there exist an open bounded convex set Ω_f in \mathbb{R}^2 , $p \in \Omega_f$, and three constants $a > 0$, $b > 0$ and $r_0 > 0$, such that for $r > r_0$ one has

$$\frac{1}{a}f(r) - b \leq \text{Vol}_\Omega(B_\Omega(p, r)) \leq (af(r) + b) \times r.$$

Therefore there are convex sets Ω with maximal entropy and zero centro-projective area, and there are convex sets with zero entropy which are not polytopes.

This corollary directly follows from our proof of the main theorem (see section 5) and Schneider and Wieacker [SW81] results on the approximability in dimension 2. The last statement follows from our work [Ver09], where we showed that polytopes have polynomial growth of order r^2 in dimension two.

The equalities and inequalities also implies the following four new results,

Corollary. *Let $\Omega \in \mathbb{R}^n$ be a convex body,*

- $d_H \leq \text{Ent}(\Omega)$, where d_H is the Hausdorff dimension of the farthest points of Ω .
- if $n = 2$ or 3 ,
 - $a(\Omega)$ is a projective invariant of Ω .
 - Let Ω^* be the polar dual of Ω , then $\text{Ent}(\Omega) = \text{Ent}(\Omega^*)$.
- if $n = 2$, then $a(\Omega) \leq \frac{1}{3-d}$.

1. NOTATION AND DEFINITIONS

A *proper* open set in \mathbb{R}^n is a set not containing a whole line.

A Hilbert geometry (Ω, d_Ω) is a non empty *proper* open convex set Ω in \mathbb{R}^n (that we shall call *convex domain*) with the Hilbert distance d_Ω defined as follows: for any distinct points p and q in Ω , the line passing through p and q meets the boundary $\partial\Omega$ of Ω at two points a and b , such that one walking on the line goes consecutively by a, p, q, b . Then we define

$$d_\Omega(p, q) = \frac{1}{2} \ln[a, p, q, b],$$

where $[a, p, q, b]$ is the cross ratio of (a, p, q, b) , i.e.,

$$[a, p, q, b] = \frac{\|q - a\|}{\|p - a\|} \times \frac{\|p - b\|}{\|q - b\|} > 1,$$

with $\|\cdot\|$ the canonical euclidean norm in \mathbb{R}^n . If either a or b is at infinity the corresponding ratio will be taken equal to 1.

Note that the invariance of the cross ratio by a projective map implies the invariance of d_Ω by such a map.

These geometries are naturally endowed with a C^0 Finsler metric F_Ω as follows: if $p \in \Omega$ and $v \in T_p\Omega = \mathbb{R}^n$ with $v \neq 0$, the straight line passing by p and directed by v meets $\partial\Omega$ at two points p_Ω^+ and p_Ω^- . Then let t^+ and t^- be two positive numbers such that $p + t^+v = p_\Omega^+$ and $p - t^-v = p_\Omega^-$, in other words these numbers corresponds to the time necessary to reach the boundary starting at p with the speed v

and $-v$. Then we define

$$F_\Omega(p, v) = \frac{1}{2} \left(\frac{1}{t^+} + \frac{1}{t^-} \right) \quad \text{and} \quad F_\Omega(p, 0) = 0.$$

Should p_Ω^+ or p_Ω^- be at infinity, then corresponding ratio will be taken equal to 0.

The Hilbert distance d_Ω is the length distance associated to F_Ω . We shall denote by $B_\Omega(p, r)$ the metric ball of radius r centred at the point $p \in \Omega$ and by $S_\Omega(p, r)$ the corresponding metric sphere.

Thanks to that Finsler metric, we can built two important Borel measures Ω .

The first one is called the *Busemann* volume, will be denoted by Vol_Ω (It is actually the Hausdorff measure associated to the metric space (Ω, d_Ω) , see [BBI01], exemple 5.5.13), and is defined as follows. To any $p \in \Omega$, let $\beta_\Omega(p) = \{v \in \mathbb{R}^n \mid F_\Omega(p, v) < 1\}$ be the open unit ball in $T_p\Omega = \mathbb{R}^n$ of the norm $F_\Omega(p, \cdot)$ and ω_n the euclidean volume of the open unit ball of the standard euclidean space \mathbb{R}^n . Consider the (density) function $h_\Omega: \Omega \rightarrow \mathbb{R}$ given by $h_\Omega(p) = \omega_n / \text{Leb}(\beta_\Omega(p))$, where Leb is the canonical Lebesgue measure of \mathbb{R}^n equal to 1 on the unit "hypercube".

$$\text{Vol}_\Omega(A) = \int_A h_\Omega(p) d\text{Leb}(p)$$

for any Borel set A of Ω .

The second one, called the *Holmes-Thompson* volume will be denoted by $\mu_{HT, \Omega}$, and is defined as follows. Let $\beta_\Omega^*(p)$ be the polar dual of $\beta_\Omega(p)$ and $h_{HT, \Omega}: \Omega \rightarrow \mathbb{R}$ the density defined by $h_{HT, \Omega}(p) = \text{Leb}(\beta_\Omega^*(p)) / \omega_n$. Then $\mu_{HT, \Omega}$ is the measure associated to that density.

We can actually consider a wider family of measure as follows Let \mathcal{E}_n be the set of pointed properly open convex sets in \mathbb{R}^n . These are the pairs (ω, x) , such that ω is a properly open convex set and x a point inside ω . We shall say that a function $f: \mathcal{E}_n \rightarrow \mathbb{R}^+ \setminus \{0\}$ is a *proper density* if it is

Continuous: with respect to the Hausdorff pointed topology on \mathcal{E}_n ;

Monotone decreasing: with respect to inclusion of the convex sets, i.e., if $x \in \omega \subset \Omega$ then $f(\Omega, x) \leq f(\omega, x)$.

Chain rule compatible: if for any projective transformation T one has

$$f(T(\omega), T(x)) \text{Jac}(T) = f(\omega, x).$$

We will say that f is a *normalised proper density* if in addition f coincides with the standard Riemannian volume on the Hyperbolic geometry of ellipsoids. Let us denote by PD_n the set of proper densities over \mathcal{E}_n .

Let us now recall a result of Benzecri [Ben60] which states that the action of the group of projective transformations on \mathcal{E}_n is co-compact. Then, as remarked by L. Marquis, for any pair f, g of proper densities, there exists a constant $C > 0$ ($C \geq 1$ for the normalised ones) such that for any $(\omega, x) \in \mathcal{E}$ one has

$$(6) \quad \frac{1}{C} \leq \frac{f(\omega, x)}{g(\omega, x)} \leq C.$$

In the same way we defined the Busemann and the Holmes-Thompson volumes, to any proper density f one can associate a Borel measure on Ω $\mu_{f,\Omega}$. Integrating the equivalence (6) we obtain that for any pair f, g of densities, there exists a constant $C > 0$ such that for any Borel set $U \subset \Omega$ we will have

$$(7) \quad \frac{1}{C} \mu_{g,\Omega}(U) \leq \mu_{f,\Omega}(U) \leq C \mu_{g,\Omega}(U).$$

We shall call *proper measures with density* the family of measures obtain this way.

To a proper density $g \in PD_{n-1}$ we can also associate a $n - 1$ -dimensional measure, denoted by $\mu_{n-1,g,\Omega}$, on hypersurfaces in Ω as follows. Let S_{n-1} be smooth a hypersurface, and consider for a point p in the hypersurface S_{n-1} its tangent hyperplane $H(p)$, then the measure will be given by

$$(8) \quad d\mu_{n-1,g,\Omega}(p) = d\mu_{g,\Omega \cap H(p)}(p).$$

In section 5 we will simply denote by $\text{Vol}_{n-1,\Omega}$ the $n - 1$ -dimensional measure associated with the Holmes-Thompson measure.

Let now $\mu_{f,\Omega}$ be a proper measure with density over Ω , then the volume entropy of Ω is defined by

$$(9) \quad \text{Ent}(\Omega) = \liminf_{r \rightarrow +\infty} \frac{\ln \mu_{f,\Omega}(B_\Omega(p, r))}{r}.$$

This number does not depend on either f or p and is furthermore equal to the spherical entropy (see [BBV10]):

$$(10) \quad \text{Ent}(\Omega) = \liminf_{r \rightarrow +\infty} \frac{\ln \mu_{n-1,g,\Omega}(S_\Omega(p, r))}{r}.$$

2. CRITICAL EXPONENT AND VOLUME ENTROPY

We show the equality between the critical exponent of any discretisation with the volume entropy, which therefore also holds for any discrete co-compact subgroup of isometries of a divisible Hilbert Geometry.

Definition 1 (Nets and Discretisations).

- A subset \mathcal{G} of a Hilbert geometry Ω is *separated*, if there exists $\varepsilon > 0$ such that the distance between any two distinct points of \mathcal{G} is greater than or equal to ε .

- A discrete subset \mathcal{G} of a Hilbert geometry Ω is a *net* if there exists $\eta > 0$, such that any point of Ω is at distance less or equal to η from a point of \mathcal{G} . Such an η is called a covering radius of the net.
- A separated net will be called a *discretisation*.

Remark 2. Our present definition of a discretisation is slightly different from our paper [Ver09]. In the later we called discretisation the graph structure associated to a separated net. Notice also that a maximal separated set is a net and a minimal net is separated.

Definition 3 (Critical exponent, Poincaré series). Let \mathcal{G} be a discrete subset of a convex set (Ω, d_Ω) endowed with its Hilbert Geometry and $x_0 \in \Omega$ some fixed point.

- We call *Poincaré Series* of \mathcal{G} , the following series

$$\sum_{\gamma \in \mathcal{G}} e^{-sd_\Omega(\gamma, x_0)}.$$

- We call *critical exponent* of \mathcal{G} , denoted by $\delta_{\mathcal{G}}$, the number

$$\delta_{\mathcal{G}} = \inf \left\{ s > 0 \left| \sum_{\gamma \in \mathcal{G}} e^{-sd_\Omega(\gamma, x_0)} < +\infty \right. \right\}.$$

- If Γ is a discrete sub-group of isometries of (Ω, d_Ω) , its critical exponent δ_Γ , will be the critical exponent of Γx_0 , the orbit of the point x_0 .

Notice that the definition of the critical exponent does not depend on the base point x_0 .

The following property is classical. We nevertheless give its proof by sake of completeness.

Property 4. Let \mathcal{G} be a discrete subset of a convex set (Ω, d_Ω) endowed with its Hilbert Geometry and $x_0 \in \Omega$ some fixed point. Then we have the following equality

$$\delta_{\mathcal{G}} = \liminf_{R \rightarrow +\infty} \frac{\ln \# \{ \gamma \in \mathcal{G} \mid d(x_0, \gamma) \leq R \}}{R}.$$

Proof. Let us define $N(R)$ by

$$N(R) = \# \{ \gamma \in \mathcal{G} \mid d(x_0, \gamma) \leq R \}$$

and $h(\mathcal{G})$ by

$$h(\mathcal{G}) = \liminf_{R \rightarrow +\infty} \frac{\ln N(R)}{R}$$

In the annulus $K < d(x_0, y) \leq K + 1$ there are $N(K + 1) - N(K)$ points of \mathcal{G} , thus

$$(11) \quad (N(K+1) - N(K))e^{-s(K+1)} \leq \sum_{K < d(x_0, \gamma) \leq K+1} e^{-sd_\Omega(\gamma, x_0)} \leq N(K+1)e^{-sK}.$$

Suppose $s > h(\mathcal{G})$, and fix some ε such that $0 < \varepsilon < s - h(\mathcal{G})$. Then for K large enough there is a constant C such that $N(K+1)e^{-sK} \leq Ce^{-\varepsilon K}$. Hence the partial sums $\sum_{K < d(x_o, \gamma) \leq K+1} e^{-sd_\Omega(\gamma, x_o)}$ are bounded by the term of a converging series and thus the Poincaré series of \mathcal{G} converges and we deduce that $\delta_{\mathcal{G}} \leq h(\mathcal{G})$.

Now suppose $h(\mathcal{G}) > 0$, and let $0 < s < h(\mathcal{G})$. Then by the left inequality of equation 11 we have

$$(12) \quad N(K+1)e^{-s(K+1)} \leq N(K+1)e^{-s(K+1)} + (1 - e^{-s}) \sum_{\kappa=0}^K N(\kappa)e^{-s\kappa} \leq \sum_{d(x_o, \gamma) \leq K+1} e^{-sd_\Omega(\gamma, x_o)}.$$

As a consequence if we fix some ε such that $0 < \varepsilon < h(\mathcal{G}) - s$, then for K large enough there is a constant C such that $N(K+1)e^{-s(K+1)} > Ce^{\varepsilon K}$, which implies that the poincaré series of \mathcal{G} diverges. Hence $\delta_{\mathcal{G}} \geq h(\mathcal{G})$. \square

Proposition 5. *Let (Ω, d_Ω) be a convex set endowed with its Hilbert Geometry*

- (1) *The critical exponent of a separated set is less than or equal to the volume entropy of Ω .*
- (2) *The critical exponent of a net is greater than or equal to the volume entropy of Ω .*
- (3) *The critical exponent of any discretisation is equal to the volume entropy.*

Proof. Let \mathcal{G} be an ε -separated set in (Ω, d_Ω) . Let V_ε the lower bound on the volume of balls of radius $\varepsilon/2$ as given in [CV07], then in the ball of radius R centred at x_o , denoted by $B(x_o, R)$, there are at most $\text{Vol}_\Omega(B(x_o, R + \varepsilon/2))/V_\varepsilon$ points of \mathcal{G} . Thus

$$(13) \quad \#\{\gamma \in \mathcal{G} \mid d(x_o, \gamma) \leq R\} \leq \text{Vol}_\Omega(B(x_o, R))/V_\varepsilon$$

and our first claim follows from property 4.

Let \mathcal{G} be a net in (Ω, d_Ω) with covering radius η . Let V^η be the upper bound of the volume of balls of radius η as given in [CV07]. Then we have

$$(14) \quad \text{Vol}_\Omega(B(x_o, R)) \leq \#\{\gamma \in \mathcal{G} \mid d(x_o, \gamma) \leq R\} \cdot V^\eta$$

and again our second claims follows from property 4. \square

The following is thus a straightforward consequence.

Corollary 6. *Let (Ω, d_Ω) be a convex set endowed with its Hilbert Geometry. The critical exponent of any discrete co-compact subgroup of isometries is equal to the volume entropy.*

3. INTRINSIC AND EXTRINSIC HAUSDORFF TOPOLOGIES OF HILBERT GEOMETRIES

We compare the Hausdorff topology induced by an euclidean metric with the Hausdorff topology induced by the Hilbert metric on compact subset of an open convex set.

We recall that the Lowner ellipsoid of a compact set, is the ellipsoid with least volume containing that set. In this section we will suppose, without loss of generality, that Ω is a bounded open convex set, whose Lowner ellipsoid \mathcal{E} is the euclidean unit ball and o is the center of that ball. It is a standard result that $(1/n)\mathcal{E}$ is then contained in Ω , i.e., we have the following sequence of inclusions

$$(15) \quad \frac{1}{n}\mathcal{E} \subset \Omega \subset \mathcal{E}$$

We call *asymptotic ball* of radius R centred at o the image of Ω by the dilation of ratio $\tanh R$ centred at o , and we denote it by $AsB(o, R)$.

Let us denote by *Hausdorff-Euclidean* distance the usual Hausdorff distance between compact subset of the n -dimensional Euclidean space, and by *Hausdorff-Hilbert* the metric between compact subsets of Ω , defined in the same way as the usual Hausdorff metric, but by replacing the Euclidean distance by the Hilbert distance in the definition.

We would like to relate the Hausdorff-Hilbert neighborhoods of the asymptotic ball $AsB(o, R)$ with its Hausdorff-Euclidean neighborhoods.

Proposition 7. *Let Ω be a convex domain and let o be the centre of its Lowner ellipsoid, which is supposed to be the unit euclidean ball.*

- (1) *The $(1 - \tanh(R))/2n$ -Hausdorff-Euclidean neighborhood of the asymptotic ball $AsB(o, R)$ is contained in its $((\ln 3)/2)$ -Hausdorff-Hilbert neighborhood.*
- (2) *For any $K > 0$, the K -Hausdorff-Hilbert neighborhood of the asymptotic ball $AsB(o, R)$ is contained in its $(1 - \tanh(R))$ -Hausdorff-Euclidean neighborhood.*

Proof. For any point $p \in \partial\Omega$ on the boundary of Ω , and for $0 < t < 1$ let $\varphi_t(p) = o + t \cdot \overrightarrow{op}$. This map sends $\partial\Omega$ bijectively on the boundary of $AsB(o, \operatorname{arctanh} t)$, which we shall denote by $AsS(0, \operatorname{arctanh} t)$ and call the *asymptotic sphere* centred at o with radius $\operatorname{arctanh} t$.

First claim:

Any point of a compact set in the $(1 - \tanh(R))/2n$ -Hausdorff-Euclidean neighborhood of $AsB(o, R)$, either lies inside $AsB(o, R)$, or is contained in an euclidean ball of radius $(1 - \tanh(R))/2n$ centred on a point of $AsB(o, R)$.

We recall that the ball of radius $1/n$, i.e., $1/n\mathcal{E}$ is inside Ω , and thus so is the ball of radius $1/2n$. Let $p \in \partial\Omega$ be a point of the boundary, by

convexity the interior of the convex closure of p and $1/n\mathcal{E}$ is a subset of Ω . It is the projection of a cone of basis $1/n\mathcal{E}$. Hence the image of $1/n\mathcal{E}$ by the dilation of ratio $0 < \alpha < 1$ centred at p lies in that "cone", therefore stays inside Ω , and it is an euclidean ball of radius α/n centred at $\varphi_{1-\alpha}(p)$, that we shall denote by $\mathcal{E}_{p,\alpha}$. For the Hilbert distance of $\mathcal{E}_{p,\alpha}$ a point in the euclidean ball of radius $\alpha/2n$ centred at $\varphi_{1-\alpha}(p)$ is at a distance less or equal to $1/2 \ln 3$ from $\varphi_{1-\alpha}(p)$. Now a standard comparison arguments states that for any two points x and y in $\mathcal{E}_{p,\alpha} \subset \Omega$ the following inequality occurs

$$d_{\Omega}(x, y) \leq d_{\mathcal{E}_{p,\alpha}}(x, y).$$

From this inequality it follows that any point in the euclidean ball of radius $\alpha/2n$ centred at $\varphi_{1-\alpha}(p)$, is inside $B_{\Omega}(\varphi_{1-\alpha}(p), 1/2 \ln 3)$, the Hilbert metric ball centred at $\varphi_{1-\alpha}(p)$. Now for any $1 \geq \alpha > 1 - \tanh R$, the ball of radius $\alpha/2n$ contains the ball of radius $(1 - \tanh R)/2n$.

This implies that for any point in $AsB(o, R)$, the euclidean ball of radius $(1 - \tanh R)/2n$ centred at that point is contained in the Hilbert ball of radius $1/2 \ln 3$ centred at the same point, which allows us to obtain the first part of our claim.

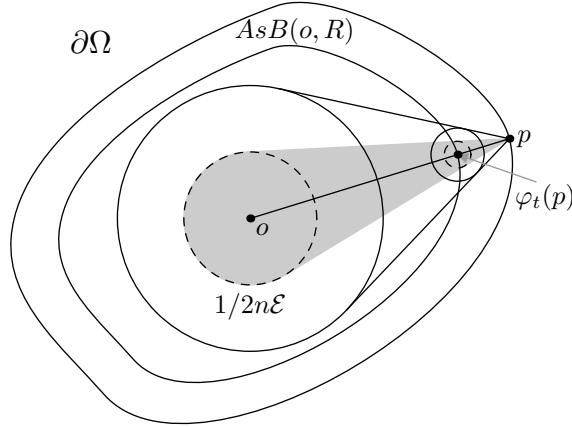


FIGURE 1. Illustration of Proposition 7's proof

Second claim: This follows from the fact that under our assumptions, Ω itself is in the $(1 - \tanh R)$ Hausdorff-Euclidean neighborhood of $AsB(o, R)$. \square

Corollary 8. *Let Ω be a convex domain and let o be the centre of its Lowner ellipsoid, which is supposed to be the unit euclidean ball.*

- (1) *The $(1 - \tanh(R + \ln 2))/2n$ -Hausdorff-Euclidean neighborhood of $B(o, R)$ is contained in its $\ln(3(n + 1))$ -Hausdorff-Hilbert neighborhood.*

- (2) For any $K > 0$, the K -Hausdorff-Hilbert neighborhood of $B(o, R)$ is contained in its $\left(1 - \tanh(R + K - \ln(n + 1))\right)$ -Hausdorff-Euclidean neighborhood.

Proof. This follows from the fact that for any Hilbert geometry, under our assumptions, the following inclusions are satisfied:

$$(16) \quad \begin{aligned} B(o, R) &\subset AsB(o, R + \ln 2), \text{ and} \\ AsB(o, R) &\subset B(o, R + \ln(n + 1)). \end{aligned}$$

This is a refinement of a result of [CV04] in our case.

Let x be a point on the boundary and let x^* be the other intersection of the straight line (ox) with $\partial\Omega$. Then following our assumption we have

$$(17) \quad \begin{aligned} 1/2n < 1/n &\leq xo \leq 1 \\ 1/2n < 1/n &\leq ox^* \leq 1 \end{aligned}$$

Actually the first inclusion is always true. Indeed suppose y is on the half line $[ox)$ such that $d_\Omega(o, y) \leq R$ which in other words implies that we have

$$\frac{ox}{yx} \frac{yx^*}{ox^*} \leq e^{2R}$$

therefore

$$ox \leq e^{2R} \frac{ox^*}{yx^*} (ox - oy) \leq e^{2R} (ox - oy)$$

which implies in turn that

$$oy \leq \frac{e^{2R} - 1}{e^{2R}} ox \leq (1 - e^{-2R}) ox \leq \tanh(R + \ln 2) ox.$$

Now regarding the second inclusion, let again y be on the half line $[ox)$ but this time such that $oy \leq \tanh(R) ox$, then on one hand we have

$$\frac{ox}{yx} = \frac{ox}{ox - oy} \leq \frac{1}{1 - \tanh(R)} = \frac{e^{2R} + 1}{2}.$$

and on the second hand thanks to the inequalities (17) we get

$$(18) \quad \frac{yx^*}{ox^*} \leq \frac{ox + ox^*}{ox^*} \leq 1 + \frac{ox}{ox^*} \leq 1 + n,$$

which implies that

$$(19) \quad \frac{ox}{yx} \frac{yx^*}{ox^*} \leq \frac{e^{2R} + 1}{2} (1 + n) \leq (1 + n) e^{2R} \leq (1 + n)^2 e^{2R}.$$

□

Remark 9. Notice that proposition 7 is still valid if we only assume Ω to contain a ball of radius $1/n$ centred at o . In the same way, corollary 8 and the inclusions (16) will still be true if Ω contains the ball of radius $1/2n$ centred at o and is included in the ball of radius 1 centred

at the same point o , because one needs only to replace n by $2n$ in the inequalities (18) and (19) and remark that $(1 + 2n) \leq (1 + n)^2$.

4. POLYTOPAL DIMENSION AND APPROXIMABILITY OF CONVEX BODIES

We define the polytopal dimension of a convex body and investigate its properties. Noticeably we recall that an upper bound exists which is attained by the convex C^2 bodies.

Definition 10. Let Ω be an open convex set in \mathbb{R}^n . For any $\varepsilon > 0$, let $N(\varepsilon, \Omega)$ be the smallest number of vertices of a polytope whose Hausdorff distance to Ω is less than ε .

We call *polytopal dimension* of Ω , denoted by $PD(\Omega)$, the number

$$PD(\Omega) = 2 \liminf_{\varepsilon \rightarrow 0} \frac{\ln N(\varepsilon, \Omega)}{-\ln \varepsilon}$$

The following result is due to R. Schneider and J. A. Wieacker [SW81]

Theorem 11 ([SW81]). Let $a_s(\Omega) := \liminf_{\varepsilon \rightarrow 0+} N(\varepsilon, \Omega)\varepsilon^s$, then $s \rightarrow a_s$ admits a critical value $\alpha(\Omega)$ called *approximability number* of Ω , such that, if $s > \alpha(\Omega)$ then $a_s(\Omega) = 0$ and if $s < \alpha(\Omega)$ then $a_s(\Omega) = \infty$.

It is quite obvious that the following occurs

Proposition 12. The polytopal dimension is twice the approximability number, i.e., $PD(\Omega) = 2\alpha(\Omega)$.

Proof. Let us take a sequence of real numbers (ε_i) converging to 0 such that,

$$PD(\Omega) = 2 \lim_{i \rightarrow +\infty} \frac{\ln N(\varepsilon_i, \Omega)}{-\ln \varepsilon_i}.$$

Let us begin by assuming that $\alpha(\Omega) > s$. According to theorem 11, for any $M > 0$ and i large enough $N(\varepsilon_i)\varepsilon_i^s > e^M$. In other words for i large enough

$$e^{s \cdot \ln(\varepsilon_i) + \ln N(\varepsilon_i, \Omega)} > e^M,$$

taking the logarithm on both sides and then dividing by $\ln(\varepsilon_i)$, which is negative, implies that

$$s < \frac{M}{\ln(\varepsilon_i)} + \frac{\ln N(\varepsilon_i, \Omega)}{-\ln(\varepsilon_i)},$$

and as $i \rightarrow +\infty$ this gives $2s \leq PD(\Omega)$, which implies that

$$(20) \quad PD(\Omega) \geq 2\alpha(\Omega).$$

Now let us assume that $\alpha(\Omega) < s$. Once more, following theorem 11, for any $M < 0$ and i large enough $N(\varepsilon_i, \Omega)\varepsilon_i^s < e^{-M}$. In other words for i large enough

$$e^{s \cdot \ln(\varepsilon_i) + \ln N(\varepsilon_i, \Omega)} < e^{-M},$$

taking the logarithm on both sides and then dividing by $\ln(\varepsilon_i)$ which is negative, implies that

$$s > \frac{-M}{\ln(\varepsilon_i)} + \frac{\ln N(\varepsilon_i, \Omega)}{-\ln(\varepsilon_i)},$$

and as $i \rightarrow +\infty$ this gives $2s \geq PD(\Omega)$, which in turn implies

$$(21) \quad PD(\Omega) \leq 2\alpha(\Omega).$$

Equations 20 and 21 imply our statement. \square

Now the main result by E. M. Bronshteyn and L. D. Ivanov [BI76] asserts that for any convex set Ω inscribed in the unit euclidean ball, there are no more than $c(n)\varepsilon^{(1-n)/2}$ points whose convex hull is no more than ε away from Ω in the Hausdorff topology. This easily induces the following upper bound for any convex set Ω :

$$PD(\Omega) \leq n - 1$$

5. VOLUME ENTROPY AND POLYTOPAL DIMENSION

We bound from below the volume entropy by the approximability in all dimensions. In dimension two and three we prove the inverse inequality and in doing so prove the upper bound entropy conjecture and the equality between these two invariants in these two dimensions.

Theorem 13. *Let Ω be a bounded convex domain in \mathbb{R}^2 or \mathbb{R}^3 . The polytopal dimension of Ω is bigger than the volume entropy, i.e.,*

$$\text{Ent}(\Omega) \leq PD(\Omega)$$

To prove that theorem, we will actually prove the following stronger statement, valid in dimension 2 and 3, which should be true in higher dimension as well.

Theorem 14. *Let us choose a family of proper measures with density denoted by Vol_* . Then for any $n = 2$ or 3 there are affine maps a_n, b_n from $\mathbb{R} \rightarrow \mathbb{R}$ and polynomials p_n, q_{n-1} of degree n and $n - 1$ such that for any open convex polytope \mathcal{P}_N with N vertices inside the unit euclidean ball of \mathbb{R}^n containing the ball of radius $1/2n$, one has*

$$(22) \quad \begin{aligned} \text{Vol}_{\mathcal{P}_N} B_{\mathcal{P}_N}(o, R) &\leq a_n(N)p_n(R) \\ \text{Vol}_{n-1, \mathcal{P}_N} S_{\mathcal{P}_N}(o, R) &\leq b_n(N)q_{n-1}(R). \end{aligned}$$

The same result holds for the asymptotic balls.

Proof of theorem 14. We will consider the Holmes-Thompson measure for the proof and work with the asymptotic balls. A change of family of measures will only change the constants. The passage from the asymptotic balls to the metric balls is done thanks to the inclusions (16).

Let P_R be the asymptotic ball of radius R centred at o , and us denote by $c_n = \ln(n + 1)$.

Two dimensional case: The length of each edge of P_R in \mathcal{P}_N is less than $2(R + c_2)$ thanks to the triangular inequality and because by the proof of corollary 8, P_R is inside the Hilbert ball of radius $R + c_2$ centred at o of \mathcal{P}_R . Therefore the length of the boundary of P_R is less than $N \cdot 2(R + c_n)$.

Now using the co-area inequality (lemma 2.13) of [BBV10] we conclude that the area of P_R is less than $C_2 \times N \times (R^2 + c_2 R)$, where C_2 is a constant depending on the dimension.

Three dimensional case: Consider one of the faces of P_R , then by minimality of the Holmes-Thompson volume, the area of that face is less than the sum of the areas of the triangles obtained thanks to the convex closure of o and an edge of the given face of P_R .

Claim: *the area of such a triangle is less than $C(R + c_3)^2$, for C some constant independent of R .*

Thanks to that claim, whose proof we postpone, if $e(N)$ is the number of edges of \mathcal{P}_N , the area of ∂P_R is less than $e(N)C(R + c_3)^2$. Let $f(N)$ be the number of faces of \mathcal{P}_N and let us recall Euler's formula:

$$N - e(N) + f(N) = 2,$$

besides, as each face is surrounded by at least three edges and each edge belongs to two faces, one has the classical inequality (where equality is obtained in a simplex),

$$3f(N) \leq 2e(N).$$

Combining the previous two inequalities we get a linear upper bound of the number of edges by the number of vertexes as follows:

$$2 \leq N - (1/3)e(N) \Rightarrow e(N) \leq 3N - 6.$$

Hence the area of ∂P_R is less than $(3N - 6)C(R + c_3)^2$, and by using once again the co-area inequality [BBV10] we conclude that the volume of P_R is less than $C' \times (3N - 6)((R + c_3)^3 - c_3^3)$, where C' is a constant independent of the three dimensional convex domain.

Proof of the Claim: In substance we claim that in a polygon \mathcal{P} , if p and q are two consecutive vertexes, and p_ρ, q_ρ are two points at a distance less than ρ from a given point o , on the lines (op) and (oq) respectively, then the affine triangle $op_\rho q_\rho$ has an area less than $C\rho^2$, for some universal constant C . In aim to convince the reader, let p'

and q' be, respectively, the second intersection of the lines (op) and (oq) with the convex $\partial\mathcal{P}$.

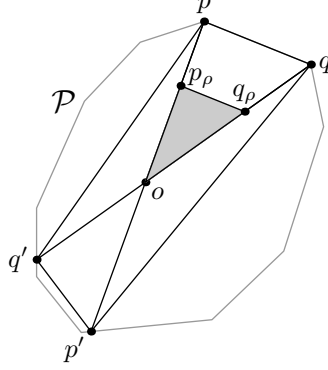


FIGURE 2. The area of the triangle (op_ρ, q_ρ) is bounded by $C\rho^2$.

Then the volume of our triangle for \mathcal{P} is less than its volume in the quadrilateral $(pp'q'q')$, while the distances op_ρ and oq_ρ remain the same by construction. Up to a change of chart, we can suppose that this quadrilateral is actually a square. This allows us to use the result in Vernicos [Ver11] which states that the Hilbert geometry of the square is bi-lipschitz to the product of the Hilbert geometries of its sides, using the identity as a map. Therefore our affine triangle is inside the disc of radius $C\rho$ for this product geometry, for some constant C independent of our initial conditions, which implies that its area for Ω is less than $C'\rho^2$ for some universal constant. \square

Let us remark that if we link this to our study of the asymptotic volume of the Hilbert geometry of polytopes [Ver12] we obtain the following corollary

Corollary 15. *Let \mathcal{P}_N be an open convex polytope with N vertices in \mathbb{R}^n , for $n = 2$ or 3 , then there are three constants α_n , β_n and γ_n such that for any point $p \in \mathcal{P}_N$ one has*

$$\alpha_n \times N \leq \liminf_{R \rightarrow +\infty} \frac{\text{Vol}_{\mathcal{P}_N} B_{\mathcal{P}_N}(p, R)}{R^n} \leq \beta_n \times N + \gamma_n$$

Now let us come back to our initial problem and see how theorem 14 implies theorem 13.

Proof of theorem 13. We remind the reader that $\text{Vol}_{n-1, \Omega}$ stands for the $n-1$ -dimensional Holmes-Thompson measure. Let o be the centre of the Lowner ellipsoid of Ω which is supposed to be the unit euclidean ball. We consider R large enough such that the euclidean ball of radius $1/2n$ is inside all the convex studied in the sequel.

Consider the asymptotic ball $AsB(o, R)$ and let P_R the polygon inside $AsB(o, R)$ at euclidean-Hausdorff distance less than $1 - \tanh(R)/2n$,

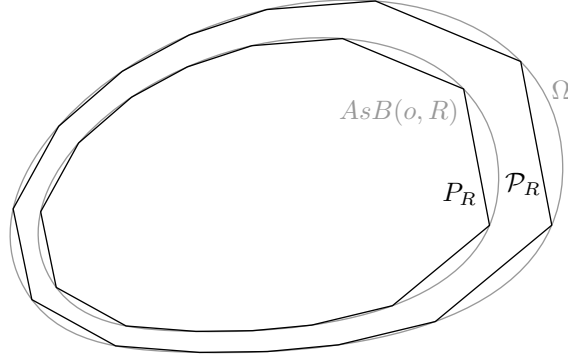


FIGURE 3. The asymptotic ball and an approximating polytope

with the minimum number $N(R) := N\left(\Omega, \frac{1-\tanh(R)}{2n \tanh(R)}\right)$ of vertices. By proposition 7 it lies in its $\ln 3/2$ Hilbert-Hausdorff neighborhood, and therefore contains the asymptotic ball of radius $R - C$, for C a constant independent of R (*e.g.* one can take $C = \ln(n + 1) + \ln 6$ following (i) and (ii) of the proof of corollary 8).

Thanks to the Crofton formula (see [AF98, BBV10, CV07]) we know that the area of P_R is less than the area of the asymptotic ball $AsB(o, R)$, but bigger than the area of the asymptotic ball of radius $R - C$.

Therefore one has

$$\text{Vol}_{n-1, \Omega}(AsB(o, R - C)) \leq \text{Vol}_{n-1, \Omega}(P_R) \leq \text{Vol}_{n-1, \Omega}(AsB(o, R))$$

which implies that the logarithm of the areas of P_R and $AsB(o, R)$ are asymptotically the same in the following sense

$$\lim_{R \rightarrow +\infty} \frac{\ln \text{Vol}_{n-1, \Omega}(AsB(o, R))}{\ln \text{Vol}_{n-1, \Omega}(P_R)} = 1.$$

If we denote by \mathcal{P}_R the image of P_R by the dilation of ratio $1/(\tanh R)$, then by construction, \mathcal{P}_R is inside Ω and therefore we have

$$\text{Vol}_{n-1, \Omega}(P_R) \leq \text{Vol}_{n-1, \mathcal{P}_R}(P_R).$$

Now let us use the result from theorem 14, for R such that $\tanh(R) > 3/4$ we have the existence of two constants a_n, b_n and a polynomial Q_n of degree n such that

$$\text{Vol}_{n-1, \Omega}(P_R) \leq (a_n N(R) + b_n) Q_n(R)$$

Now

$$\frac{\ln(N(R))}{R} = 2 \frac{\ln(N(R))}{\ln(e^{2R})} = 2 \frac{\ln(N(R))}{\ln(1 - \tanh R) + K} \frac{\ln(1 - \tanh(R)) + K}{\ln(e^{2R})}$$

which implies that

$$\liminf_{R \rightarrow +\infty} \frac{\ln(N(R))}{R} = PD(\Omega).$$

□

Corollary 16. *Let Ω be an open bounded convex set in \mathbb{R}^n , for $n = 2$ or 3 , then*

$$\text{Ent}(\Omega) \leq n - 1.$$

We are now going to study the inverse inequality.

Theorem 17. *Let Ω be a bounded convex domain in \mathbb{R}^n . The volume entropy is bigger or equal to the polytopal dimension of Ω , i.e.,*

$$PD(\Omega) \leq \text{Ent}(\Omega).$$

Lemma 18. *Consider four points a, b, c and d such that the straight lines D_{ab} , D_{bc} and D_{cd} are distinct and the scalar products $\langle ab, bc \rangle$ and $\langle bc, cd \rangle$ are strictly positive. Let also denote by q the intersection point between the straight lines (ab) and (cd) .*

We also suppose that Ω is a convex domain such that the segments ab , bc and cd belong to its boundary.

Let p be a point inside the convex Ω and denote by p' the intersection between the straight line (pq) and the segment bc .

Finally we denote by $b(R)c(R)$ the image under the dilation centred at p with ratio $0 < \tanh(R) < 1$ of the segment bc .

Then one has

$$(23) \quad d_{\Omega}(b(R), c(R)) \geq \frac{1}{2} \ln \left(\frac{bc}{s \cdot BC} \frac{\tanh(R)}{1 - \tanh(R)} + 1 \right) \\ + \frac{1}{2} \ln \left(\frac{bc}{(1-s) \cdot BC} \frac{\tanh(R)}{1 - \tanh(R)} + 1 \right)$$

where BC is the image of bc under the dilation centred at q sending p' on p and s is the ratio bp'/bc .

Proof. Straightforward computation, using the fact that the convex domain Ω is inside the convex obtained as the intersection of the half planes defined by the line (ab) , (bc) and (cd) . \square

Proof of Theorem 17. Without loss of generality we suppose that the euclidean unit ball is the Lowner ellipsoid of Ω , and that o is the centre of that ball.

We will do two proofs, one in the two dimensional case, which gives more informations and leads to the left hand side of the intermediate growth corollary, and one for all dimension which consists in building an appropriate separated net.

The two dimensional case:

We will consider a family (\mathcal{P}_R) of polytopes as follows.

- For each R , \mathcal{P}_R contains $AsB(o, R)$ and is in the $(1 - \tanh(R))/4$ -Hausdorff-Euclidean neighborhood of the asymptotic ball $AsB(o, R)$,

which is included in the $(\ln 3)/2$ -Hausdorff-Hilbert neighborhood by proposition 7. Therefore, \mathcal{P}_R will lie inside Ω .

- All but one edge of \mathcal{P}_R are tangent to $AsB(o, R)$ and all its vertexes belong to the boundary of its $\epsilon(R) = 1 - \tanh(R)/4$ -Hausdorff neighborhood (let us denote it $\partial_R AsB$).

To see this, start from any point x in $\partial_R AsB$ and follow this algorithm:

- Step 1* draw one of the tangent to $AsB(o, R)$, it will meet the boundary of its $\epsilon(R)$ at two points x_1 and x_2 , where $\vec{o}x_1, \vec{o}x_2$ are positively oriented.
- Step 2* We begin again at x_2 drawing the other tangent to $AsB(o, R)$ passing by x_2 , which will meet the boundary at x_3 .
- Step 3* for $k > 2$, if the second tangent from x_k has its second intersection with $\partial_R AsB$ on the arc of from x_1 to x_k (in the orientation of the construction), we stop and consider for \mathcal{P}_R the convex hull of x_1, \dots, x_k , otherwise we take for x_{k+1} this other side of this tangent and start again that step.

Now this algorithm will necessarily finish, because the arclength of $x_i x_{i+1}$ on $\partial_R AsB(o, R)$ built this way is bigger than $2\epsilon(R)$, by convexity. At the end of this algorithm we obtain, by minimality, a polygon which has at least $\tilde{N}(R) = N(1 - \tanh(R)/4, AsB(o, R))$.

By the inclusion of [CV04] we obtain

$$B(o, R - 1) \subset AsB(o, R) \subset \mathcal{P}_R \subset B(o, R + n + (\ln 3)/2)$$

which implies that the behaviour of $\ln(\text{Vol}_\Omega \mathcal{P}_R)/R$ is the same as the behaviour of $\ln(\text{Vol}_\Omega B(o, R))/R$ as $R \rightarrow +\infty$.

Now let \mathfrak{P}_R be the image of \mathcal{P}_R under the dilation of ratio $\tanh(R)^{-1}$ centred at o . By construction \mathfrak{P}_R contains Ω , which implies

$$\text{Vol}_{\mathfrak{P}_R} \mathcal{P}_R \leq \text{Vol}_\Omega \mathcal{P}_R.$$

Claim: Let $b(R)$ and $c(R)$ be the two points of tangency of two incident edges of \mathcal{P}_R at I , tangent to $AsB(o, R)$. Then there is a constant $M > 0$ and $R_0 > 0$ depending only on R_0 such that for any $R > R_0$,

$$d_\Omega(b(R), c(R)) \geq d_{\mathfrak{P}_R}(b(R), c(R)) \geq M.$$

Let us assume the claim is satisfied and for $R > R_0$ consider a vertex v of \mathcal{P}_R whose incident edges are tangent to $AsB(o, R)$. Let b and c the two points of tangency, then by the triangle inequality, $bv + cv \geq M$. Therefore the length of \mathcal{P}_R is bigger than $(N(R) - 2)M$, where $N(R)$ is number of edges of \mathcal{P}_R . (because of the possible exception at x_1 and the last point of the construction above).

Hence the "volume entropy" of the family \mathcal{P}_R is bigger than

$$\lim_{k \rightarrow +\infty} \frac{\ln(N(R))}{R}$$

and as we have

$$\begin{aligned}
 \frac{\ln(\tilde{N}(R))}{R} &= 2 \frac{\ln(\tilde{N}(R))}{-\ln e^{-2R}} \\
 &= 2 \frac{\ln\left(N\left(\frac{1-\tanh R}{2n}, AsB(o, R)\right)\right)}{-\ln e^{-2R}} \\
 &= 2 \frac{\ln\left(N\left(\frac{1-\tanh R}{2n \tanh R}, \Omega\right)\right)}{-\ln e^{-2R}} \\
 (24) \quad &= 2 \frac{\ln\left(N\left(\frac{1-\tanh R}{2n \tanh R}, \Omega\right)\right)}{-\ln\left(\frac{1-\tanh R}{2n \tanh R}\right)} \times \frac{\ln\left(\frac{1-\tanh R}{2n \tanh R}\right)}{\ln e^{-2R}}.
 \end{aligned}$$

we deduce from that the desired result by taking the infimum limit as $R \rightarrow +\infty$

Let us now prove the claim. To do so, let $a(R)$ (resp. $d(R)$) be opposite vertex to I on the edge containing $b(R)$ (resp. $c(R)$).

Now let us consider the image I, a, b, c and d of the five points $I(R), a(R), b(R), c(R)$ and $d(R)$ by the dilation of ratio $1/\tanh R$ centred at o . Then we are in the same configuration as in lemma 18, with \mathfrak{P}_R instead of Ω . Let $u(R) = \frac{bc}{BC} \frac{\tanh(R)}{1-\tanh(R)}$, then following (23) we have

$$d_{\mathfrak{P}_R}(b(R), c(R)) \geq \frac{1}{2} \ln\left(1 + \frac{u(R) + u(R)^2}{s(1-s)}\right).$$

Therefore we need to obtain a lower bound for $u(R)$. To do this, let p be the intersection of the line oI with the lines (bc) . Then thanks to Thales's theorem we have

$$\frac{BC}{bc} = \frac{oI}{pI} = \frac{op + pI}{pI} = 1 + \frac{op}{pI}$$

Concerning the distance op , recall that the unit ball centred at o is the Lowner ellipsoid of Ω and therefore we get $op \leq \frac{1}{\tanh(R)}$, because by convexity p is in Ω .

Regarding the distance pI , as $I(R)$ is on the boundary of the $(1 - \tanh(R))/4$ euclidean neighborhood of $AsB(o, R)$, I is on the boundary of the $(1 - \tanh(R))/4 \tanh(R)$ neighborhood of Ω . Hence we obtain $pI \geq (1 - \tanh(R))/4 \tanh(R)$, because the segment $[p, I]$ intersects Ω . This way we obtain

$$\frac{BC}{bc} \leq 1 + \frac{4}{1 - \tanh(R)}$$

which in turn implies that

$$1 \leq \frac{5 - \tanh(R)}{1 - \tanh(R)} \frac{bc}{BC} \leq \frac{5}{1 - \tanh(R)} \frac{bc}{BC}.$$

Hence

$$(25) \quad \frac{\tanh(R)}{5} \leq u(R)$$

Therefore if $\tanh(R_0) = 1/2$ then for all $R > R_0$ we get $10u(R) > 1$.

Finally using the fact that $s(1-s) \leq 1/4$ and taking $R > R_0$ we get

$$d_{\mathfrak{P}_R}(b(R), c(R)) \geq \frac{1}{2} \ln \left(1 + \frac{2}{5} + \frac{1}{25} \right).$$

General case:

We suppose now that our convex is n -dimensional. Consider

$$V(R) = B(o, R + (\ln 3)/2)$$

the $(\ln 3)/2$ -Hilbert neighborhood of the metric ball $B(o, R)$, and take a maximal $\delta = (\ln 3)/4$ -separated net \mathcal{S}_R on its boundary. This set contains $N_1(R)$ points.

Now let us take the convex hull \mathcal{C}_R of these points. This is a polytope with $N_2(R) \leq N_1(R)$ vertices.

Let us prove that it is included in the 2δ -Hilbert neighborhood of $B(o, R)$ and contains $B(o, R)$. If this hold we will have for some real constant c independent of R (see corollary 8 once again),

$$N_2(R) \geq \tilde{N}(R - c) = N(1 - \tanh(R - c)/4, AsB(o, R - c)).$$

First notice that $V(R)$ is a convex set (see Busemann [Bus55], chapter II, section 18, page 105). Therefore the convex hull is inside $V(R)$.

Now let us suppose by contradiction that \mathcal{C}_R does not contain $B(o, R)$. Hence there exists some points q in $B(o, R)$ which is not in \mathcal{C}_R . Therefore, by the Hahn-Banach separation theorem, there exists a linear form a , some constant x and hyperplane $H = \{x \mid a(x) = c\}$ which separates q and \mathcal{C}_R , i.e., $a(q) > c$ and $a(x) < c$ for all $x \in \mathcal{C}_R$. Consider then $H_q = \{x \mid a(x) = a(q)\}$ the parallel hyperplane to H containing q . Let us say that a point x such that $a(x) \geq a(q)$ is *above* the hyperplane H_q .

Then let us define by $V'_o = \{x \in \partial V(R) \mid a(x) \geq a(q)\}$ the part of the boundary of $V(R)$ which is above H_q .

Now we want to metrically project each point of V'_o onto H_q , that is to say that to each point of V'_o we associate its closest point on H_q . However if Ω is not strictly convex, the projection might not be unique (see the appendix A), that is why we are going to distinguish two cases.

First case: The convex set Ω is strictly convex, then the metric projection is a map from V'_o to H_q and it is continuous, furthermore the point on $H_q \cap V'_o$ are fixed and by convexity $H_q \cap V'_o$ is homeomorphic to a $n - 2$ -dimensional sphere. Therefore by Borsuk-Ulam's theorem (or its version known as the *antipodal map theorem*), there is a point p on V'_o whose metric projection is q .

Now as p is on the boundary of $V(R)$ necessarily

$$d_\Omega(p, q) \geq (\ln 3)/2.$$

hence for all points x in $H_q \cap V'_o$, we have

$$d_\Omega(p, x) \geq d_\Omega(p, q) \geq (\ln 3)/2.$$

Second case: The convex set Ω is not strictly convex. Then let us approximate it by a smooth and strictly convex set Ω' such that $\Omega \subset \Omega'$, and for all pair of points $x, y \in V(R)$,

$$(26) \quad \frac{2}{3} \times d_{\Omega'}(x, y) \geq d_\Omega(x, y) \geq d_{\Omega'}(x, y).$$

Then metrically project V'_o onto H_q with respect to Ω' . By the same argument from the first case, we obtain a point p such that for all x in $H_q \cap V'_o$ we have

$$d_{\Omega'}(p, x) \geq d_{\Omega'}(p, q) \geq \frac{3}{2}d_\Omega(p, q) \geq \frac{3}{4}(\ln 3)$$

which also implies by 26 that for all x in $H_q \cap V'_o$ we have

$$d_\Omega(p, x) \geq 3(\ln 3)/4.$$

In either cases, using the lemma 24 of the appendix B, we deduce that all points on ∂V_R at distance less or equal to $(\ln 3)/4$ from p are above H_q and are therefore contained in V'_o . We then infer that there are no points of S_R at distance less or equal to $(\ln 3)/4$ from p , which contradicts the maximality of the set S_R .

We can now conclude, by considering the discrete family

$$\mathcal{F} = \bigcup_{n \geq 1} S_{n \ln 3}$$

which is by our construction an $\ln 3/4$ separated set, and therefore its critical exponent is a lower bound of the entropy following 5.

For any $R \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n \ln 3 \leq R < (n+1) \ln 3$, we thus get the following inequality:

$$\begin{aligned} \#\{x \in \mathcal{F} \mid d(o, x) \leq R\} &\geq \#\{x \in \mathcal{F} \mid d(o, x) \leq n \ln 3\} \\ &\geq \#S_{n \ln 3} \geq \tilde{N}(n \ln 3 - c) \end{aligned}$$

which implies that

$$\delta_{\mathcal{F}} \geq \liminf_{n \rightarrow \infty} \frac{\ln \tilde{N}(n \ln 3 - c)}{(n+1) \ln 3} \geq \liminf_{R \rightarrow \infty} \frac{\ln \tilde{N}(R)}{R} = PD(\Omega).$$

The last inequality comes from the same computations done during two dimensional proof. \square

A point x of a convex body K is called a *farthest point* of K if and only if, for some point $y \in \mathbb{R}^n$, x is farthest from y among the points of K . The set of farthest points of K , which are special exposed points, will be denoted by $\exp^* K$. Thus a point $x \in K$ belongs to $\exp^* K$ if

and only if there exists a ball which circumscribes K and contains x in its boundary.

we thus have the following corollary in dimension 2,

Corollary 19. *Let Ω be a plane Hilbert geometry, and let d_M be the Minkowski dimension of extremal points and d_H the Hausdorff dimension of the set $\exp^*\Omega$ of farthest points then we have the following inequalities*

$$(27) \quad d_H \leq \underline{\text{Ent}}(\Omega) \leq \frac{2}{3 - d_M}.$$

The left hand side inequality remains valid for higher dimensional Hilbert geometries.

Proof. The left hand side of inequality (27) comes from R. Schneider and J. A. Wieacker [SW81], whereas the right hand one from G. Berck, A. Bernig and C. Vernicos [BBV10]. \square

Remark 20. Inequality (27) induces a new result concerning the approximability in dimension 2, as it implies that

$$a(\Omega) \leq \frac{1}{3 - d}.$$

Lastly we are also able to prove the following result which relates the entropy of a convex set and the entropy of its polar body.

Corollary 21. *Let Ω be a Hilbert geometry of dimension 2 or 3, then*

$$\text{Ent}(\Omega) = \text{Ent}(\Omega^*)$$

Proof. It suffices to prove that the approximability of a convex body Ω containing the origin and its polar Ω^* are equal. However, notice that for ε small enough, if P_k is a polytope with k vertexes inside the ε -Hausdorff neighborhood of Ω , then its polar P_k^* is a polytope with k faces containing Ω^* and contained in its ε -Hausdorff neighborhood. A known fact (see Gruber [Gru07]) states that the approximability can be computed either by minimising the vertexes or the faces. Hence $a(\Omega) = a(\Omega^*)$ and our result follows from the Main Theorem. \square

APPENDIX A. METRIC PROJECTION IN A HILBERT GEOMETRY

The following is a reformulation and a detailed proof of a statement found in section 21 and 28 of Busemann-Kelly's book [BK53] in any dimension.

Proposition 22. *Let (Ω, d_Ω) be a Hilbert geometry in \mathbb{R}^n . Let p be a point of Ω and H an hyperplane intersecting Ω . Then $q \in H \cap \Omega$ is a metric projection of p onto H , i.e.,*

$$d_\Omega(p, H) = d_\Omega(p, q),$$

if and only if $\partial\Omega$ has, at its intersection with the straight line (pq) , supporting hyperplanes concurrent with H (the intersection of these three hyperplanes is an $n - 2$ -dimensional affine space).

Proof. Let us suppose first that such concurrent support hyperplanes exists. Let x and y be the intersections of the line (pq) with $\partial\Omega$. Assume that ξ and η are supporting hyperplanes of $\partial\Omega$ respectively at x and y whose intersection with H is the $n - 2$ -affine space W . Let us show that for any $p' \in (pq)$ and any $q' \in H$ we have

$$(28) \quad d_{\Omega}(p', q') \geq d_{\Omega}(p', q).$$

Let us suppose that x is on the half line $[qp')$ and y on the half line $[p'q)$ and denote by x' and y' the intersection of $\partial\Omega$ with the half line $[q'p')$ and $[p'q')$ respectively. Then let x_0 be the intersection of ξ with the line $(p'q')$ and y_0 the intersection of $(p'q')$ with η . By Thales' theorem, the cross-ratio of $[x_0, p', q', y_0]$ is equal to the cross ratio of $[x, p', q, y]$ and standard computation shows that $[x_0, p', q', y_0] \leq [x', p', q', y']$, with equality if and only if $x_0 = x'$ and $y' = y_0$. Hence the inequality (28) holds, and if the convex set is strictly convex, this inequality is always strict, for $q' \neq q$.

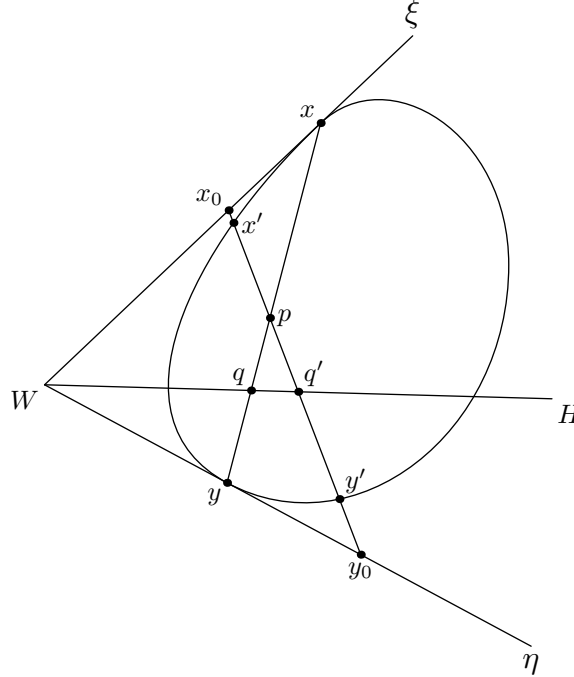


FIGURE 4. Metric projection of p on H .

Reciprocally: recall that when a point q' of Ω goes to the boundary, its distance to p goes to infinity. Hence by continuity of the distance and compactness there exists a point q on $H \cap \Omega$ such that $d_{\Omega}(p, H) = d_{\Omega}(p, q)$. Now consider the Hilbert ball $B_{\Omega}(p, r)$ of radius $r = d_{\Omega}(p, H)$

centred at p . Let once more x , y , ξ and η be defined as before, and let H' be the hyperplane passing by q and $\xi \cap \eta = W$. Then this hyperplane has to be tangent to the ball $B_\Omega(p, r)$, otherwise one can find a point q' on H' inside the open ball (i.e. $d(p, q') < r$), however by the reasoning done in our first step we would conclude that this point is at a distance bigger or equal to r , which would be a contradiction. By minimality of the point q , H is also a supporting hyperplane of $B_\Omega(p, r)$ at q . Hence we have to distinguish between two cases. If Ω is C^1 , then by uniqueness of the tangent hyperplanes at every point $H = H'$. Otherwise, Ω is not C^1 at x or y . In that case it is possible to change one of the hyperplane, say ξ , with ξ' passing by x and $H \cap \eta$ (which might be at infinity, which would mean that we consider parallel hyperplanes). \square

Notice that there is no uniqueness of the metric projections (also called "foot" by Busemann). However if the convex set is strictly convex, then we will have a unique projection, if furthermore the convex is C^1 , this projection will be given by a unique pair of supporting hyperplanes.

APPENDIX B. DISTANCE FUNCTION TO A SPHERE IN A HILBERT GEOMETRY

This appendix is an adaptation of a proof in a Minkowski space provided to the author by A. Thompson [Tom]

Let us first start by recalling the following important fact regarding the distance of a point to a geodesic in a Hilbert geometry (see Busemann [Bus55], chapter II, section 18, page 109):

Proposition 23. *Let (Ω, d_Ω) be a Hilbert Geometry. The distance function of a geodesic to a point is a peakless function, i.e., if $\gamma: [t_1, t_2] \rightarrow \Omega$ is a geodesic segment, then for any $x \in \Omega$ and $t_1 \leq s \leq t_2$ one has*

$$d_\Omega(x, \gamma(s)) \leq \max \left\{ d_\Omega(x, \gamma(t_1)), d_\Omega(x, \gamma(t_2)) \right\}.$$

Let us now turn our attention to metric spheres in a two dimensional Hilbert geometry.

Proposition 24. *Let (Ω, d_Ω) be a two dimensional Hilbert Geometry. Suppose o is point of Ω , and p and q are two points on the intersection of the metric sphere $S(o, R)$ centred at o and radius R with a line passing by o . If C denotes one of the arcs of the sphere $S(o, R)$ from p to q , then for any point p' on the half line $[o, p)$, the function $\varphi(x) = d_\Omega(p', x)$ is monotonic on C .*

Proof. Let p, x, y, q be points on that order on C . We have to show that

$$d_{\Omega}(p', x) \leq d_{\Omega}(p', y).$$

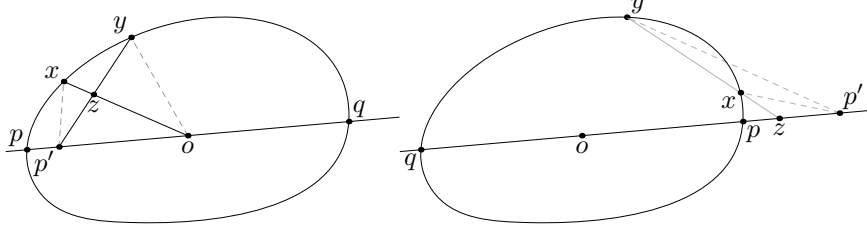


FIGURE 5. Monotonicity of the distance of a point to a sphere

Suppose first that the line segments $[o, x]$ and $[p', y]$ intersect at a point z . Hence we have

$$\begin{aligned} d_{\Omega}(o, x) + d_{\Omega}(p', y) &= (d_{\Omega}(o, z) + d_{\Omega}(z, x)) + (d_{\Omega}(p', z) + d_{\Omega}(z, y)) \\ &= (d_{\Omega}(p', z) + d_{\Omega}(z, x)) + (d_{\Omega}(o, z) + d_{\Omega}(z, y)) \\ &\geq d_{\Omega}(p', x) + d_{\Omega}(o, y). \end{aligned}$$

now, as $d_{\Omega}(o, y) = d_{\Omega}(o, x) = R$, the result follows.

Suppose now that $[o, x]$ and $[p', y]$ do not intersect, which implies that p' is outside the ball $B(o, R)$. Then the line (yx) intersects (op) at z . Because x and y lie on the sphere of radius R , $d_{\Omega}(o, z) > R$. Also, as p is one of the nearest points to p' on C , we have $d_{\Omega}(p', z) \leq d_{\Omega}(p', p) \leq d_{\Omega}(p', y)$. Hence if apply the proposition 23 to the segment $[z, y]$ and p' , as $x \in [z, y]$ we get

$$d_{\Omega}(p', x) \leq \max\{d_{\Omega}(p', z), d_{\Omega}(p', y)\} = d_{\Omega}(p', y).$$

□

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